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On the Cauchy Problem of the Vlasov-Poisson-BGK System: Global Existence of Weak Solutions

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Abstract Global-in-time existence of weak solutions to the Cauchy problem of the three dimensional Vlasov-Poisson-BGK system is shown for initial data belonging to the space $L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ with $p > 9$ and having finite second order velocity moments. This result solves partially the well-posed problem for the Vlasov-Poisson-BGK system proposed by B. Perthame: "Higher moments for kinetic equations: the Vlasov-Poisson and Fokker-Planck cases," Math. Meth. Appl. Sci. 13:441–452, 1990.

Keywords Vlasov-Poisson-BGK system · Cauchy problem · Weak solution · Global existence

1 Introduction

We consider a statistical mechanics system (e.g., a stellar dynamical system or a plasma) which occupies the three dimensional Euclid space $\mathbb{R}^3$ and consists of a large number of particles interacting through long range forces as well as short range ones. Let $f(t,x,\xi) \geq 0$ be the microscopic density of particles in this system at time $t \geq 0$ and position $x \in \mathbb{R}^3$, moving with velocity $\xi \in \mathbb{R}^3$, and let $\rho_f(t,x)$, $u_f(t,x)$ and $\theta_f(t,x)$ be respectively the mass (charge) density, bulk velocity and temperature (i.e., macroscopic quantities) of the system at time t and position x, that is to say that [6]

$$\begin{pmatrix} \rho_f \\ \rho_f u_f \\ \rho_f |u_f|^2 + 3\rho_f \theta_f \end{pmatrix} (t,x) = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \\ |\xi|^2 \end{pmatrix} f(t,x,\xi) d\xi, \quad t \geq 0,\ x \in \mathbb{R}^3. \tag{1.1}$$

It is well known that the long range interactions among particles are self consistent and can be characterized by the macroscopic density $\rho_f(t,x)$ through Poisson's equation [11].

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On the other hand, for a dilute system such as a stellar dynamical system or a plasma, the short range interactions between particles are assumed to be binary and can be described by several nonlinear operators, one of which is the BGK collision operator [5, 6]. In this situation, the evolution of the system is governed by the Vlasov-Poisson-BGK system

$$\begin{aligned}&\partial_t f + \xi\cdot\nabla_x f + E(t,x)\cdot\nabla_\xi f = J(f), \quad (x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3,\ t>0,\\ &f(0,x,\xi) = f_0(x,\xi), \quad (x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3,\end{aligned} \tag{1.2}$$

$$-\Delta_x U(t,x) = \gamma\rho_f(t,x), \qquad \lim_{|x|\to\infty} U(t,x) = 0, \quad x\in\mathbb{R}^3,\ t>0, \tag{1.3}$$

$$E(t,x) = -\nabla_x U(t,x), \quad x\in\mathbb{R}^3,\ t>0. \tag{1.4}$$

Here $J(f) = M[f](t,x,\xi) - f(t,x,\xi)$ is the BGK collision operator with

$$M[f](t,x,\xi) = \frac{\rho_f(t,x)}{(2\pi\theta_f(t,x))^{N/2}}\exp\left(-\frac{|\xi - u_f(t,x)|^2}{2\theta_f(t,x)}\right) \tag{1.5}$$

being the local Maxwellian corresponding to the unknown $f(t,x,\xi)$. Obviously, the BGK operator has five collision invariants 1, ξ and $|\xi|^2$, that is to say that we have for any reasonable microscopic density f

$$\int_{\mathbb{R}^3} J(f)(1,\xi,|\xi|^2)d\xi = 0.$$

Functions $U(t,x)$ and $E(t,x)$ in (1.3) and (1.4) are respectively the potential and force field for long range interactions for which $\gamma = \mp 1$ corresponds to gravitational or repulsive cases (e.g., $\gamma = -1$ for a stellar dynamical system and $\gamma = 1$ for a plasma). The nonnegative function $f_0(x,\xi)$ in (1.2) is an initial microscopic density of the system and is assumed to be known.

The Vlasov-Poisson-BGK system (1.1)–(1.5) can be viewed as a higher order correction to the collisionless Vlasov-Poisson system, it is also a kinetic model of the Vlasov-Poisson-Boltzmann system. For the classical Vlasov-Poisson system, there is a numerous literature devoted to the studies of it (see e.g., [2–4, 9, 11, 14–17, 19, 20, 25, 26, 28, 29] and the references therein), and the Vlasov-Poisson-Boltzmann system also attracts several researchers' attentions (see e.g., [1, 7, 8, 12, 13, 18, 22, 30, 31] and the references therein). It is well known that the BGK collision operator is a good approximation of the Boltzmann operator and plays an important role in numerical simulations [6]. On the other hand, no rigorous results on the Vlasov-Poisson-BGK system have been obtained so far. Actually, an open problem on solvability of its Cauchy problem was posed by B. Perthame in 1990 [24], and it still remains unsolved. The present paper is devoted to investigating this problem and establishing certain mathematical results on the existence of global weak solutions.

Another important issue for the Vlasov-Poisson-BGK system (1.1)–(1.5) is the uniqueness of its solutions. Although this problem has been well solved for the classical Vlasov-Poisson system (see e.g., [19] and [20]), from the author's point of view it is a rather difficult problem even for more regular solution classes in the Vlasov-Poisson-BGK situation.

Before we proceed further, we first note that it follows from the classical potential theory that for sufficiently regular macroscopic density ρ_f (this is the case in this paper), the unique solution of (1.3) and (1.4) can be expressed by

$$U(t,\cdot) = -\gamma\Gamma(\cdot)\star_x\rho_f(t,\cdot), \quad E(t,\cdot) = \gamma K(\cdot)\star_x\rho_f(t,\cdot), \tag{1.6}$$

where $\Gamma(x) = -\frac{1}{4\pi|x|}$ is the fundamental solution of the Laplacian Δ_x in $\mathbb{R}^3$ and $K(x) = \nabla_x \Gamma(x) = \frac{x}{4\pi|x|^3}$. Hence, the Vlasov-Poisson-BGK system (1.1)–(1.5) is equivalent to (1.1), (1.2), (1.5) and (1.6).

As has been mentioned above, this paper is aimed at establishing global-in-time solutions to the Cauchy problem (1.1)–(1.5). So, we first of all fix the notation of weak solutions to the kinetic equations (1.1)–(1.5) used in the present paper. A nonnegative function $f(t,x,\xi) \in L^1([0,T); L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ is said to be a weak solution on $[0,T)$ to the system (1.1)–(1.5) if $E(t,x)$ verifies (1.6) and

$$\int_0^T dt \int_{\mathbb{R}^3\times\mathbb{R}^3} f\left(\partial_t\phi + \xi\cdot\nabla_x\phi + E\cdot\nabla_\xi\phi\right)dxd\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\phi|_{t=0}dxd\xi$$
$$-\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} J(f)\phi dxd\xi = 0 \tag{1.7}$$

for any test function $\phi(t,x,\xi) \in C_c^\infty([0,T)\times\mathbb{R}^3\times\mathbb{R}^3))$. If in addition $f(t,x,\xi) \in L^1_{\text{loc}}([0,\infty); L^1(\mathbb{R}^3\times\mathbb{R}^3))$, and if (1.6) and (1.7) are valid for all $T>0$, then f is said to be a global weak solution to the system (1.1)–(1.5). Due to the fact that only first order derivatives are involved in (1.7), we can replace the test function space $C_c^\infty([0,T)\times\mathbb{R}^3\times\mathbb{R}^3))$ by $C_c^1([0,T)\times\mathbb{R}^3\times\mathbb{R}^3))$.

Suppose that $f(t,x,\xi)$ is a weak solution to the system (1.1)–(1.5), its kinetic energy and potential energy at time t are respectively defined by (see, e.g., [14–16] and [25])

$$\mathcal{E}_k(f)(t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^2 f(t,x,\xi)dxd\xi \quad \text{and} \quad \mathcal{E}_p(f)(t) = \gamma\int_{\mathbb{R}^3}|E(t,x)|^2dx. \tag{1.8}$$

Integration by parts, we obtain another expression of $\mathcal{E}_p(f)(t)$:

$$\mathcal{E}_p(f)(t) = -\gamma\int_{\mathbb{R}^3\times\mathbb{R}^3}\Gamma(x-y)\rho_f(t,x)\rho_f(t,y)dxdy.$$

For $p\in[1,\infty]$, the norm of $f\in L^p(\mathbb{R}^3\times\mathbb{R}^3)$ is denoted by $\|f\|_p$. With the above notations, we can describe our main results.

Theorem 1.1 *Suppose that the initial value $f_0(x,\xi)$ is a nonnegative function satisfying*

$$(1+|\xi|^2)f_0 \in L^1(\mathbb{R}^3\times\mathbb{R}^3), \quad f_0\in L^p(\mathbb{R}^3\times\mathbb{R}^3) \tag{1.9}$$

for $p>9$, then there exists a global weak solution $f(t,x,\xi)$ to the system (1.1)–(1.5) such that

$$\|f(t)\|_1 = \|f_0\|_1, \qquad \|f(t)\|_p \le \exp(C_pt)\|f_0\|_p, \quad t\ge 0, \tag{1.10}$$

where $C_p>0$ is a constant depending only upon p. Moreover, we have

$$\mathcal{E}_k(f)(t) + \mathcal{E}_p(f)(t) \le \mathcal{E}_k(f_0) + \mathcal{E}_p(f_0), \quad t\ge 0. \tag{1.11}$$

Consequently, there exists a positive constant $M = M(\|f_0\|_1, \|f_0\|_p, p)$ such that

$$\mathcal{E}_k(f)(t) \le 2[\mathcal{E}_k(f_0) + |\mathcal{E}_p(f_0)|] + M\exp(2C_pt), \quad t\ge 0. \tag{1.12}$$

Remark 1.1

(1) In (1.12), $|\mathcal{E}_p(f_0)|$ can be estimated by generalized Young's inequality, Hölder's inequality and interpolation inequality in Lemma 2.2 (1) of Sect. 2 as follows:

$$
\begin{aligned}
|\mathcal{E}_p(f_0)| &= \int_{\mathbb{R}^3} |E(0,x)|^2 dx \\
&\le \int_{\mathbb{R}^3} [|K(\cdot)| \star_x \rho_{f_0}(\cdot)]^2 dx \le C_1 \|\rho_{f_0}\|_{6/5}^2 \\
&\le C_1 \|\rho_{f_0}\|_1^{\frac{7p-9}{6(p-1)}} \|\rho_{f_0}\|_{r(p)}^{\frac{5p-3}{6(p-1)}} \\
&\le C_1 \|\rho_{f_0}\|_1^{\frac{7p-9}{6(p-1)}} \left(C \|f_0\|_p^{\frac{2p}{5p-3}} \||\xi|^2 f_0\|_1^{\frac{3p-3}{5p-3}} \right)^{\frac{5p-3}{6(p-1)}} \\
&\le C' \|f_0\|_1^{\frac{7p-9}{6(p-1)}} \|f_0\|_p^{\frac{p}{3(p-1)}} \||\xi|^2 f_0\|_1^{\frac{1}{2}} < \infty,
\end{aligned}
$$

where

$$
r(p) = \begin{cases} \frac{5p-3}{3p-1}, & 1 < p < \infty, \\ \frac{5}{3}, & p = \infty. \end{cases}
$$

This inequality and (1.12) indicate that the kinetic energy $\mathcal{E}_k(f)(t)$ is locally bounded on $[0,\infty)$ and grows at most exponentially as $t \to \infty$.

(2) In the case of $\gamma = 1$, we have $\mathcal{E}_p(f)(t) = \int_{\mathbb{R}^3} |E(t,x)|^2 dx \ge 0$. Then (1.11) implies that $\mathcal{E}_k(f)(t) \le \mathcal{E}_k(f_0) + \mathcal{E}_p(f_0)$ for all $t \ge 0$. Hence, the kinetic energy $\mathcal{E}_k(f)(t)$ is globally bounded on $[0,\infty)$.

(3) Compared with the classical results on global existence of weak solutions for the Vlasov-Poisson system obtained by E. Horst and R. Hunze in [16], Theorem 1.1 requires higher integrability for the initial datum f_0, i.e., we assume $p > 9$ rather than $p > (12 + 3\sqrt{5})/11$. This point is due to the following facts.

To prove the sequence of approximate solutions f^ε constructed in Lemma 2.1 (see Sect. 2) converges to a weak solution f of the system (1.1)–(1.5), two difficulties must be overcome. The first is to show the weak convergence of $E^\varepsilon f^\varepsilon$ to Ef, and the second is to prove the continuity of the nonlinear operator $M[f]$, i.e., $M[f^\varepsilon] \to M[f]$ in a local L^1 sense as $\varepsilon \to 0^+$. Due to those facts, we have to establish uniform control on higher velocity moments for the sequence of approximate solutions f^ε. Precisely speaking, we have to prove that for some $\delta > 0$ small enough and any $R < \infty$, there exists a positive constant K independent of ε such that (see inequality (2.11) in Sect. 2)

$$
\int_0^T dt \int_{B(0,R)\times\mathbb{R}^3_\xi} |\xi|^{2+\delta} f^\varepsilon dx d\xi \le K.
$$

To this end, by the velocity moments lemma (Lemma 2.4), we obtain inequality (2.7) in Sect. 2. Hence, the above uniform estimate can be achieved by estimating each terms in the right hand side of (2.7). However, an estimate of the following term

$$
\int_0^T dt \int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} \left(1 + |\xi|^{(1/2+\delta)}\right) |g^\varepsilon| dx d\xi, \quad (g^\varepsilon = -E^\varepsilon \cdot f^\varepsilon)
$$

relies on the estimate of the electric field E^ε established in Lemma 2.3, which are only known to be valid when

$$p \geq \frac{24(1-\delta)+\sqrt{(7-6\delta)^2+14(28+3\delta)}}{(5-30\delta)}. \tag{1.13}$$

Since $\lim_{\delta\to 0^+} \frac{24(1-\delta)+\sqrt{(7-6\delta)^2+14(28+3\delta)}}{(5-30\delta)} = 9$, condition (1.13) is satisfied as long as $p > 9$ and $\delta \in (0, 1/6)$ is small enough.

The assumption $p > 9$ in Theorem 1.1 is probably not optimal and may be improved. However, in order to do so, new methods must be introduced.

2 Proof of the Main Results

In order to prove Theorem 1.1, for any $\varepsilon > 0$ we regularize the fundamental solution Γ and the convolutional kernel K by ([14] and [16])

$$\Gamma_\varepsilon(x) = -\frac{1}{4\pi(\varepsilon+|x|^2)^{1/2}}, \qquad K_\varepsilon(x) = \nabla_x \Gamma_\varepsilon(x) = \frac{x}{4\pi(\varepsilon+|x|^2)^{3/2}}.$$

Using these regularized quantities, we construct approximate equations of system (1.1)–(1.5) as follows

$$\partial_t f^\varepsilon + \xi\cdot\nabla_x f^\varepsilon + E^\varepsilon\cdot\nabla_\xi f^\varepsilon = J(f^\varepsilon), \quad f^\varepsilon(0,x,\xi) = f_0(x,\xi), \tag{2.1}$$

$$E^\varepsilon(t,x) = \gamma[K_\varepsilon(\cdot)\star_x \rho_{f^\varepsilon}(t,\cdot)](x), \quad (x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3,\ t>0, \tag{2.2}$$

where ρ_{f^ε} and $J(f^\varepsilon)$ are defined by (1.1) and (1.5) with f replaced by f^ε.

It is well known that for any $q \in [1, 3/2)$, $\|K_\varepsilon - K\|_{L^q(\mathbb{R}^3)} \to 0$ as $\varepsilon \to 0$ [16]. This fact enables us to show that a sequence of solutions to the initial value problem (2.1) (2.2) converges in some sense to a solution of the system (1.1)–(1.5) as $\varepsilon \to 0$, so long as some strong enough regularities for those solutions are available. Hence, a way of proving the main result can be divided into three steps: the first step is to show global existence of solutions to the approximate problem (2.1) (2.2), the second step is to establish various estimates of those approximate solutions, and the final step is to pass to the limit to obtain the desired solutions. Actually, results of the following lemma, the proof of which is postponed to the next section, can be served as the first step and part of the second step.

Lemma 2.1 *Suppose that the initial datum $f_0(x,\xi)$ satisfies $|\xi|^2 f_0 \in L^1(\mathbb{R}^3\times\mathbb{R}^3)$ and $f_0 \in L^1\cap L^p(\mathbb{R}^3\times\mathbb{R}^3)$ for $p\geq 2$, then for any $\varepsilon > 0$, the initial value problem (2.1) (2.2) has a nonnegative solution f^ε such that*

$$\|f^\varepsilon(t)\|_1 = \|f_0\|_1, \qquad \|f^\varepsilon(t)\|_p \leq \exp(\tilde{C}_p t)\|f_0\|_p, \quad t\geq 0, \tag{2.3}$$

$$\mathcal{E}_k(f^\varepsilon)(t) + \mathcal{E}_{p,\varepsilon}(f^\varepsilon)(t) = \mathcal{E}_k(f_0) + \mathcal{E}_{p,\varepsilon}(f_0), \quad t\geq 0. \tag{2.4}$$

Here, $\mathcal{E}_{p,\varepsilon}(f^\varepsilon)(t) = -\gamma\int_{\mathbb{R}^3\times\mathbb{R}^3}\Gamma^\varepsilon(x-y)\rho_{f^\varepsilon}(t,x)\rho_{f^\varepsilon}(t,y)dxdy$ and $\tilde{C}_p$ is a positive constant independent of ε. Moreover, there exists a positive constant $\tilde{M} = \tilde{M}(\|f_0\|_1, \|f_0\|_p, p)$ independent of ε such that

$$\mathcal{E}_k(f^\varepsilon)(t) \leq 2[\mathcal{E}_k(f_0) + |\mathcal{E}_{p,\varepsilon}(f_0)|] + \tilde{M}\exp(2\tilde{C}_p t), \quad t\geq 0. \tag{2.5}$$

Remark 2.1 By using the Sobolev's inequality in Lemma 2.2 (2) and a similar method in Remark 1.1, it is easy to show that

$$|\mathcal{E}_{p,\varepsilon}(f_0)| \le C' \|f_0\|_1^{\frac{7p-9}{6(p-1)}} \|f_0\|_p^{\frac{p}{3(p-1)}} \||\xi|^2 f_0\|_1^{\frac{1}{2}} < \infty,$$

where $C' > 0$ is a constant independent of $\varepsilon > 0$.

However, estimates of approximate solutions just like (2.3) and (2.4) are not enough to pass to the limit, and further estimates on the electric fields E^ε and higher order velocity moments of the approximate solutions are needed. In order to do so, we need the following lemma.

Lemma 2.2 (1) *Let $p \in (1, \infty]$, $0 \le \alpha < 2$ and $f(x, \xi) \in L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ be nonnegative such that $|\xi|^2 f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, then there exists a positive constant $C = C(p, \alpha)$ such that*

$$\left\| \int_{\mathbb{R}^3_\xi} |\xi|^\alpha f d\xi \right\|_{r(p,\alpha)} \le C \|f\|_p^{\frac{(2-\alpha)p}{5p-3}} \left(\int_{\mathbb{R}^3_x \times \mathbb{R}^3_\xi} |\xi|^2 f dx d\xi \right)^{\frac{(3+\alpha)p-3}{5p-3}},$$

with

$$r(p,\alpha) = \begin{cases} \frac{5p-3}{(3+\alpha)p-(1+\alpha)}, & 1 < p < \infty, \\ \frac{5}{(3+\alpha)}, & p = \infty. \end{cases}$$

As a consequence, we know that the macroscopic density

$$\rho_f(x) = \int_{\mathbb{R}^3} f(x, \xi) d\xi \in L^{r(p)}(\mathbb{R}^3)$$

and

$$\|\rho_f\|_{r(p)} \le C(p) \|f\|_p^{\frac{2p}{5p-3}} \||\xi|^2 f\|_1^{\frac{3p-3}{5p-3}},$$

where

$$r(p) = r(p,0) = \begin{cases} \frac{5p-3}{3p-1}, & 1 < p < \infty, \\ \frac{5}{3}, & p = \infty. \end{cases}$$

(2) (*Sobolev inequality*) *Let $p_1, p_2 \in (1, \infty)$, $\lambda \in [0, 3)$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{\lambda}{3} = 2$. Then for any $\psi_1 \in L^{p_1}(\mathbb{R}^3)$, $\psi_2 \in L^{p_2}(\mathbb{R}^3)$, we have*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\psi_1(x)||\psi_2(y)|}{|x-y|^\lambda} dx dy \le C \|\psi_1\|_{p_1} \|\psi_2\|_{p_2},$$

where $C = C(p_1, p_2, \lambda)$ is a positive constant.

For the second part (the Sobolev's inequality) of the lemma we refer the readers to Lemma 3.4 in [16] (see also [27], p. 31, Example 3). On the other hand, the interpolation inequality given in the first part of the lemma with $\alpha = 0$ is well known (see e.g., Lemma 5.5 in [15]), for the general case, we refer the readers to Lemma 1.8 in [28]. Here, for readers convenience, we sketch its proof. For any $R > 0$, we have by the Hölder's inequality (where $|\mathbb{S}^2|$ denotes the area of the unit sphere in $\mathbb{R}^3$)

$$\begin{aligned}\int_{\mathbb{R}^3_\xi} |\xi|^\alpha f d\xi &\le \int_{|\xi|\le R} |\xi|^\alpha f d\xi + \frac{1}{R^{2-\alpha}} \int_{|\xi|>R} |\xi|^2 f d\xi \\ &\le \left(\int_{|\xi|\le R} |\xi|^{\frac{\alpha p}{p-1}} d\xi\right)^{1-1/p} \left(\int_{|\xi|\le R} |f|^p d\xi\right)^{1/p} + \frac{1}{R^{2-\alpha}} \int_{|\xi|>R} |\xi|^2 f d\xi \\ &\le \left(\frac{(p-1)|\mathbb{S}^2|}{3(p-1)+\alpha p}\right)^{\frac{p-1}{p}} R^{\frac{3(p-1)+\alpha p}{p}} \left(\int_{\mathbb{R}^3_\xi} |f|^p d\xi\right)^{1/p} + \frac{1}{R^{2-\alpha}} \int_{\mathbb{R}^3_\xi} |\xi|^2 f d\xi.\end{aligned}$$

Taking

$$R = \left(\frac{3(p-1)+\alpha p}{(p-1)|\mathbb{S}^2|}\right)^{\frac{p-1}{5p-3}} \left(\frac{\int_{\mathbb{R}^3_\xi} |\xi|^2 f d\xi}{(\int_{\mathbb{R}^3_\xi} |f|^p d\xi)^{1/p}}\right)^{\frac{p}{5p-3}},$$

we obtain

$$\int_{\mathbb{R}^3_\xi} |\xi|^\alpha f d\xi \le C_1(p,\alpha) \left(\int_{\mathbb{R}^3_\xi} |\xi|^2 f d\xi\right)^{\frac{(3+\alpha)p-3}{5p-3}} \left(\int_{\mathbb{R}^3_\xi} |f|^p d\xi\right)^{\frac{2-\alpha}{5p-3}},$$

where $C_1(p,\alpha)$ is a positive constant depending only upon p and α. It follows that

$$\left(\int_{\mathbb{R}^3_\xi} |\xi|^\alpha f d\xi\right)^{r(p,\alpha)} \le C_2(p,\alpha) \left(\int_{\mathbb{R}^3_\xi} |\xi|^2 f d\xi\right)^{\frac{(3+\alpha)p-3}{(3+\alpha)p-(1+\alpha)}} \left(\int_{\mathbb{R}^3_\xi} |f|^p d\xi\right)^{\frac{2-\alpha}{(3+\alpha)p-(1+\alpha)}},$$

where $C_2(p,\alpha) = C_1(p,\alpha)^{r(p,\alpha)}$. Then integrating against x over $\mathbb{R}^3_x$ and using the Hölder's inequality once more, we obtain the desired inequality.

Based on this lemma, we can establish the following estimates on the electric fields E^ε, which plays an important role in the present paper, especially in the uniform control of higher velocity moments for approximate solutions mentioned in Remark 1.1(3).

Lemma 2.3 *Let $f(x,\xi) \in L^1 \cap L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ be nonnegative such that $|\xi|^2 f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, and let $E^\varepsilon(x) = (K^\varepsilon \star \rho_f)(x)$ and $\delta \in (0, 1/6)$ sufficiently small, then there exist positive constants $C(p)$ and $C(p,\delta)$ such that for $p \in [3,\infty]$ and $\varepsilon \ge 0$*

$$\|E^\varepsilon\|_{r'(p)} \le C(p) \|f\|_1^{\frac{5p^2-18p+9}{(p-1)(15p-9)}} \|f\|_p^{\frac{4p^2}{(p-1)(15p-9)}} \||\xi|^2 f\|_1^{\frac{2p}{(5p-3)}},$$

and for $p \ge \frac{24(1-\delta)+\sqrt{(7-6\delta)^2+14(28+3\delta)}}{(5-30\delta)}$ and $\varepsilon \ge 0$

$$\|E^\varepsilon\|_{r'(p,1/2+\delta)} \le C(p,\delta) \|f\|_1^{\frac{(1-6\delta)p-(9-6\delta)}{12(p-1)}} \|f\|_p^{\frac{p[(11+6\delta)p-(3+6\delta)]}{6(p-1)(5p-3)}} \||\xi|^2 f\|_1^{\frac{(11+6\delta)p-(3+6\delta)}{4(5p-3)}}.$$

Here $r'(p)$ and $r'(p,1/2+\delta)$ are exponents conjugate to $r(p)$ and $r(p,1/2+\delta)$ respectively.

Proof If $p \in [3, \infty]$, then $\frac{15p-9}{11p-9} \le r(p)$. By generalized Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned}\|E^\varepsilon\|_{r'(p)} &= \|K^\varepsilon \star \rho_f\|_{r'(p)} \le \left\| \frac{1}{|x|^2} \star \rho_f \right\|_{r'(p)} \\ &\le C(p)\|\rho_f\|_{\frac{15p-9}{11p-9}} \le C(p)\|f\|_1^{\frac{5p^2-18p+9}{(p-1)(15p-9)}} \|\rho_f\|_{r(p)}^{\frac{2p(5p-3)}{(p-1)(15p-9)}}.\end{aligned}$$

Using Lemma 2.2 (1) to $\|\rho_f\|_{r(p)}$, we get

$$\|E^\varepsilon\|_{r'(p)} \le C(p)\|f\|_1^{\frac{5p^2-18p+9}{(p-1)(15p-9)}} \|f\|_p^{\frac{4p^2}{(p-1)(15p-9)}} \||\xi|^2 f\|_1^{\frac{6p}{(15p-9)}}.$$

If $p \ge \frac{24(1-\delta)+\sqrt{(7-6\delta)^2+14(28+3\delta)}}{(5-30\delta)}$, then $\frac{6(5p-3)}{(19-6\delta)p-(15-6\delta)} \le r(p)$. Similarly, we have

$$\begin{aligned}\|E^\varepsilon\|_{r'(p,1/2+\delta)} &= \|K^\varepsilon \star \rho_f\|_{r'(p,1/2+\delta)} \le \left\| \frac{1}{|x|^2} \star \rho_f \right\|_{r'(p,1/2+\delta)} \\ &\le C(p,\delta)\|\rho_f\|_{\frac{6(5p-3)}{(19-6\delta)p-(15-6\delta)}} \\ &\le C(p,\delta)\|f\|_1^{\frac{(1-6\delta)p-(9-6\delta)}{12(p-1)}} \|\rho_f\|_{r(p)}^{\frac{(11+6\delta)p-(3+6\delta)}{12(p-1)}} \\ &\le C(p,\delta)\|f\|_1^{\frac{(1-6\delta)p-(9-6\delta)}{12(p-1)}} \|f\|_p^{\frac{p[(11+6\delta)p-(3+6\delta)]}{6(p-1)(5p-3)}} \||\xi|^2 f\|_1^{\frac{(11+6\delta)p-(3+6\delta)}{4(5p-3)}}. \qquad \square\end{aligned}$$

In order to show weak continuity of the nonlinear operator $M[f]$ as well as weak compactness of $E^\varepsilon f^\varepsilon$, we need another tool called velocity moments lemma established by B. Perthame [23] and improved by himself in [24]. Here, we state it as the following lemma (see, [24], Proposition 2).

Lemma 2.4 *Let $f_0(x,\xi) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $f(t,x,\xi), h(t,x,\xi) \in L^\infty_{\mathrm{loc}}([0,\infty), L^1(\mathbb{R}^3 \times \mathbb{R}^3))$, and let $g(t,x,\xi) \in L^\infty_{\mathrm{loc}}([0,\infty), L^1(\mathbb{R}^3 \times \mathbb{R}^3))^3$ such that*

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f = h + \nabla_\xi \cdot g, & \text{in } \mathcal{D}'((0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3), \\ f(0,x,\xi) = f_0(x,\xi). \end{cases}$$

Then, for any $\alpha \ge 0$, $0 < T < \infty$ and $K_x \subset\subset \mathbb{R}^3_x$, we have

$$\begin{aligned}&\int_0^T dt \int_{K_x \times \mathbb{R}^3_\xi} |\xi|^{3/2+\alpha} f \, dx d\xi \\ &\le C(1+\mathrm{diam}(K_x)) \int_0^T dt \int_{\mathbb{R}^3_x \times \mathbb{R}^3_\xi} \left[(1+|\xi|^\alpha)|g| + |\xi|^{(1/2+\alpha)}|h|\right] dx d\xi \\ &\quad + C(1+\mathrm{diam}(K_x)) \int_{\mathbb{R}^3_x \times \mathbb{R}^3_\xi} \left[|\xi|^{(1/2+\alpha)} + |\xi|^{(1+\alpha)}\right] |f_0| dx d\xi,\end{aligned}$$

where C is a positive constant and $\mathrm{diam}(K_x)$ is the diameter of the compact subset K_x.

Now, we are in a position to prove the main results in this paper.

Proof of Theorem 1.1 Firs, we establish the uniform control of higher velocity moments for approximate solutions mentioned in Remark 1.1(3). For any $\varepsilon > 0$, let f^ε be the global nonnegative solution to (2.1) (2.2) constructed in Lemma 2.1. Let $h^\varepsilon = J(f^\varepsilon)$ and $g^\varepsilon = -E^\varepsilon \cdot f^\varepsilon$, then

$$\begin{cases} \partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon = h^\varepsilon + \nabla_\xi \cdot g^\varepsilon, & \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3), \\ f^\varepsilon(0,x,\xi) = f_0(x,\xi). \end{cases} \tag{2.6}$$

Equation (2.6) and Lemma 2.4 imply that for any $T, R \in (0,\infty)$ and sufficiently small $\delta \in (0, 1/6)$ (Here, we take $K_x = B(0,R)$ and $\alpha = 1/2+\delta$ in Lemma 2.4)

$$\begin{aligned} &\int_0^T dt \int_{B(0,R)\times\mathbb{R}^3_\xi} |\xi|^{2+\delta} f^\varepsilon dx d\xi \\ &\quad \le K(1+R)\int_0^T dt \int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} [(1+|\xi|^{(1/2+\delta)})|g^\varepsilon| + |\xi|^{(1+\delta)}|h^\varepsilon|] dx d\xi \\ &\qquad + K(1+R)\int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} [|\xi|^{1+\delta} + |\xi|^{(3/2+\delta)}]|f_0| dx d\xi, \end{aligned} \tag{2.7}$$

where $K = 2C$ is a positive constant independent of ε. We estimate each term on the right hand side of (2.7) separately as follows. Firstly, we have by (1.9)

$$\begin{aligned} &\int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} [|\xi|^{1+\delta} + |\xi|^{(3/2+\delta)}]|f_0| dx d\xi \\ &\quad \le 2\int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} (1+|\xi|^2)|f_0| dx d\xi = K_0(f_0) < \infty. \end{aligned} \tag{2.8}$$

Secondly, by (2.5), Remark 2.1 and generalized Young's inequality (notice that 1 and $|\xi|^2$ are collision invariants of the BGK operator), we obtain

$$\begin{aligned} &\int_0^T dt \int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} |\xi|^{(1+\delta)}|h^\varepsilon| dx d\xi \\ &\le \int_0^T dt \int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} (1+|\xi|^2)|h^\varepsilon| dx d\xi \\ &\le 2\int_0^T dt \int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} (1+|\xi|^2)|f^\varepsilon| dx d\xi \\ &\le 2T\|f_0\|_1 + 4T[\mathcal{E}_k(f_0) + |\mathcal{E}_{p,\varepsilon}(f_0)|] + \tilde{M}\int_0^T \exp(2\tilde{C}_p t) dt \\ &\le T[2\|f_0\|_1 + 4\mathcal{E}_k(f_0) + 4C'\|f_0\|_1^{\frac{7p-9}{6(p-1)}} \|f_0\|_p^{\frac{p}{3(p-1)}} \||\xi|^2 f_0\|_1^{\frac{1}{2}} + \tilde{M}\exp(2\tilde{C}_p T)] \\ &= K_1(T,p,f_0) < \infty. \end{aligned} \tag{2.9}$$

Finally, it follows from $p > 9$ that there is a sufficiently small $\delta \in (0, 1/6)$ such that

$$p \geq \frac{24(1-\delta) + \sqrt{(7-6\delta)^2 + 14(28+3\delta)}}{(5-30\delta)}.$$

Hence, by Lemma 2.2 (1), Lemma 2.3 and Hölder's inequality, we get

$$\begin{aligned}
&\int_0^T dt \int_{\mathbb{R}_x^3\times\mathbb{R}_\xi^3} (1+|\xi|^{(1/2+\delta)})|g^\varepsilon| dx d\xi \\
&\quad\leq \int_0^T dt \int_{\mathbb{R}_x^3} |E^\varepsilon| dx \int_{\mathbb{R}_\xi^3} f^\varepsilon d\xi + \int_0^T dt \int_{\mathbb{R}_x^3} |E^\varepsilon| dx \int_{\mathbb{R}_\xi^3} |\xi|^{(1/2+\delta)} f^\varepsilon d\xi \\
&\quad\leq \int_0^T \|E^\varepsilon\|_{r'(p)} \left\| \int_{\mathbb{R}_\xi^3} f^\varepsilon d\xi \right\|_{r(p)} dt + \int_0^T \|E^\varepsilon\|_{r'(p,1/2+\delta)} \left\| \int_{\mathbb{R}_\xi^3} |\xi|^{(1/2+\delta)} f^\varepsilon d\xi \right\|_{r(p,1/2+\delta)} dt \\
&\quad\leq C \int_0^T \|E^\varepsilon\|_{r'(p)} \|f^\varepsilon\|_p^{\frac{2p}{5p-3}} \||\xi|^2 f^\varepsilon\|_1^{\frac{3p-3}{5p-3}} dt \\
&\qquad + C \int_0^T \|E^\varepsilon\|_{r'(p,1/2+\delta)} \|f^\varepsilon\|_p^{\frac{(3/2-\delta)p}{5p-3}} \||\xi|^2 f^\varepsilon\|_1^{\frac{(7/2+\delta)p-3}{5p-3}} dt \\
&\quad\leq C\|f_0\|_1^{\frac{5p^2-18p+9}{(p-1)(15p-9)}} \int_0^T \|f^\varepsilon\|_p^{\frac{11p^2-24p+9}{3(p-1)(5p-3)}} \||\xi|^2 f^\varepsilon\|_1 dt \\
&\qquad + C\|f_0\|_1^{\frac{(1-6\delta)p-(9-6\delta)}{12(p-1)}} \int_0^T \|f^\varepsilon\|_p^{\frac{2p}{3(p-1)}} \||\xi|^2 f^\varepsilon\|_1^{\frac{(25+10\delta)p-(15+6\delta)}{4(5p-3)}} dt,
\end{aligned}$$

where $C = C(p,\delta) > 0$ is a constant independent of ε. In consideration of (2.3) and (2.5), we have

$$\begin{aligned}
&\int_0^T dt \int_{\mathbb{R}_x^3\times\mathbb{R}_\xi^3} (1+|\xi|^{(1/2+\delta)})|g^\varepsilon| dx d\xi \\
&\quad\leq C(p,\delta)\|f_0\|_1^{\frac{5p^2-18p+9}{(p-1)(15p-9)}} \|f_0\|_p^{\frac{11p^2-24p+9}{3(p-1)(5p-3)}} \int_0^T \exp\left(\frac{11p^2-24p+9}{3(p-1)(5p-3)}\tilde{C}_p t\right) \\
&\qquad \times (2[\mathcal{E}_k(f_0) + |\mathcal{E}_{p,\varepsilon}(f_0)|] + \tilde{M}\exp(2\tilde{C}_p t)) dt \\
&\qquad + C(p,\delta)\|f_0\|_1^{\frac{(1-6\delta)p-(9-6\delta)}{12(p-1)}} \|f_0\|_p^{\frac{2p}{3(p-1)}} \int_0^T \exp\left(\frac{2p}{3(p-1)}\tilde{C}_p t\right) \\
&\qquad \times (2[\mathcal{E}_k(f_0) + |\mathcal{E}_{p,\varepsilon}(f_0)|] + \tilde{M}\exp(2\tilde{C}_p t))^{\frac{(25+10\delta)p-(15+6\delta)}{4(5p-3)}} dt \\
&\quad = K_2(T,p,\delta,f_0) < \infty.
\end{aligned} \tag{2.10}$$

Estimates (2.7)–(2.10) imply that for $p > 9$, there exist a sufficiently small $\delta \in (0, 1/6)$ and a positive constant $K(T,p,\delta,f_0)$ independent of $\varepsilon > 0$ such that for any $R > 0$

$$\int_0^T dt \int_{B(0,R)\times\mathbb{R}_\xi^3} |\xi|^{2+\delta} f^\varepsilon dx d\xi \leq (R+1)K(T,p,\delta,f_0). \tag{2.11}$$

On the other hand, estimates (2.3) and (2.5) imply that for any $T>0$, $\{f^\varepsilon:\varepsilon>0\}$ and $\{|\xi|^2 f^\varepsilon:\varepsilon>0\}$ are bounded in $L^\infty((0,T);L^1\cap L^p(\mathbb{R}^3\times\mathbb{R}^3))$ and $L^\infty((0,T);L^1(\mathbb{R}^3\times\mathbb{R}^3))$ respectively. Hence, there exists a nonnegative function f defined on $(0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3$ such that up to a subsequence

$$f^\varepsilon \to f \quad \text{weakly} \quad \text{in } L^r((0,T)\times\mathbb{R}^3\times\mathbb{R}^3) \quad \text{for } 1<r\le p,\ \varepsilon\to 0. \tag{2.12}$$

Let $\beta_R(\xi)\in C_c^\infty(\mathbb{R}^3)$ be the cutoff function corresponding to the ball $B(0,R)$ and let $f_R^\varepsilon=\beta_R f^\varepsilon$, then

$$\partial_t f_R^\varepsilon+\xi\cdot\nabla_x f_R^\varepsilon=g_R^\varepsilon,\quad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3), \tag{2.13}$$

where $g_R^\varepsilon=\beta_R J(f^\varepsilon)+(E^\varepsilon\cdot\nabla_\xi\beta_R)f^\varepsilon-\nabla_\xi\cdot(E^\varepsilon f_R^\varepsilon)$. By Corollary 2.3 in [33], Hölder's inequality, generalized Young's inequality and Lemma 2.2, we estimate each term in g_R^ε respectively as follows

$$\|\beta_R J(f^\varepsilon)(t)\|_2\le C_0\|f^\varepsilon(t)\|_2\le C_0\|f^\varepsilon(t)\|_1^{\frac{p-2}{2p-2}}\|f^\varepsilon(t)\|_p^{\frac{p}{2p-2}},$$

$$\begin{aligned}
\|(E^\varepsilon\cdot\nabla_\xi\beta_R)f^\varepsilon(t)\|_2&\le C_1|B(0,R)|^{\frac{(p-2)}{p}}\|f^\varepsilon(t)\|_p\|E^\varepsilon(t)\|_{\frac{2p}{p-2}}\\
&\le C_1|B(0,R)|^{\frac{(p-2)}{p}}\|f^\varepsilon(t)\|_p\||K|\star\rho_{f^\varepsilon}(t)\|_{\frac{2p}{p-2}}\\
&\le C_2|B(0,R)|^{\frac{(p-2)}{p}}\|f^\varepsilon(t)\|_p\|\rho_{f^\varepsilon}(t)\|_{\frac{6p}{5p-6}}\\
&\le C_3|B(0,R)|^{\frac{(p-2)}{p}}\|f^\varepsilon(t)\|_p\|\rho_{f^\varepsilon}(t)\|_1^{\frac{7p^2-39p+18}{12p(p-1)}}\|\rho_{f^\varepsilon}(t)\|_{r(p)}^{\frac{5p^2+27p-18}{12p(p-1)}}\\
&\le C_4|B(0,R)|^{\frac{(p-2)}{p}}\|f_0\|_1^{\frac{7p^2-39p+18}{12p(p-1)}}\|f^\varepsilon(t)\|_p^{\frac{p(35p-21)}{6(5p-3)(p-1)}}\||\xi|^2f^\varepsilon(t)\|_1^{\frac{5p^2+27p-18}{4p(5p-3)}},
\end{aligned}$$

and

$$\begin{aligned}
\|E^\varepsilon\cdot f_R^\varepsilon\|_2&\le|B(0,R)|^{\frac{(p-2)}{p}}\|f^\varepsilon(t)\|_p\|E^\varepsilon(t)\|_{\frac{2p}{p-2}}\\
&\le C_5|B(0,R)|^{\frac{(p-2)}{p}}\|f_0\|_1^{\frac{7p^2-39p+18}{12p(p-1)}}\|f^\varepsilon(t)\|_p^{\frac{p(35p-21)}{6(5p-3)(p-1)}}\||\xi|^2f^\varepsilon(t)\|_1^{\frac{5p^2+27p-18}{4p(5p-3)}}.
\end{aligned}$$

Here $C_i=C_i(p)$ $(i=0,1,\dots,5)$ are all positive constants independent of ε and R. Hence, for any given $R>0$, g_R^ε is uniformly bounded in $L^2((0,T)\times\mathbb{R}_x^3;\ H^{-1}(\mathbb{R}_\xi^3))$ with respect to $\varepsilon>0$. Using the velocity averaging lemma [10] to the linear transport equation (2.13), we deduce that for any fixed $R>0$, $\int_{B(0,R)}|\xi|^r f_R^\varepsilon(t,x,\xi)d\xi$ is uniformly bounded in $H^{1/4}((0,T)\times\mathbb{R}_x^3)$ for any fixed $r\ge 0$. Consequently, the compact imbedding theorem, the uniform bound of $\mathcal{E}_k(f^\varepsilon)(t)$ and estimate (2.11) ensure that $\int_{\mathbb{R}^3}|\xi|^r f^\varepsilon(t,x,\xi)d\xi$ is compact in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^3)$ for any fixed $r\in[0,2]$, which obviously implies the compactness of ρ_{f^ε}, $\rho_{f^\varepsilon}(|u_{f^\varepsilon}|^2+3\theta_{f^\varepsilon})$ and $\rho_{f^\varepsilon}u_{f^\varepsilon}$ in the spaces $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^3)$ and $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^3)^3$ respectively. Hence, we may assume up to a subsequence

$$\begin{cases}
\rho_{f^\varepsilon}\to\rho_f, & \text{in } L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^3),\ \varepsilon\to 0,\\
\rho_{f^\varepsilon}(|u_{f^\varepsilon}|^2+3\theta_{f^\varepsilon})\to\rho_f(|u_f|^2+3\theta_f), & \text{in } L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^3),\ \varepsilon\to 0,\\
\rho_{f^\varepsilon}u_{f^\varepsilon}\to\rho_f u_f, & \text{in } L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^3)^3,\ \varepsilon\to 0.
\end{cases} \tag{2.14}$$

From (2.12) and (2.14), we obtain by the method used in [23] and [33] that for any $R > 0$

$$J(f^\varepsilon) \to J(f), \quad \text{in } L^1((0,T) \times B(0,R) \times \mathbb{R}^3_\xi),\ \varepsilon \to 0. \tag{2.15}$$

Next, we show

$$E^\varepsilon f^\varepsilon \to Ef, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3),\ \varepsilon \to 0. \tag{2.16}$$

We notice that

$$\begin{aligned} E^\varepsilon f^\varepsilon - Ef = (K_\varepsilon \star \rho_{f^\varepsilon}) f^\varepsilon - (K \star \rho_f) f = [(K_\varepsilon - K) \star \rho_{f^\varepsilon}] f^\varepsilon \\ + [(K \star (\rho_{f^\varepsilon} - \rho_f)] f^\varepsilon + (K \star \rho_f)(f^\varepsilon - f). \end{aligned} \tag{2.17}$$

Firstly, we show

$$[(K_\varepsilon - K) \star \rho_{f^\varepsilon}] f^\varepsilon \to 0 \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3),\ \text{as } \varepsilon \to 0. \tag{2.18}$$

For any $\phi \in C_c^\infty((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, by Hölder's inequality and Young's inequality, we get

$$\begin{aligned} &\|[(K_\varepsilon - K) \star \rho_{f^\varepsilon}] f^\varepsilon \phi\|_{L^1_{t,x,\xi}} \le \|[(K_\varepsilon - K) \star \rho_{f^\varepsilon}]\phi\|_{L^2_{t,x,\xi}} \|f^\varepsilon\|_{L^2_{t,x,\xi}} \\ &\quad \le |\mathrm{supp}(\phi)|^{1/2} \|\phi\|_\infty \|[(K_\varepsilon - K) \star \rho_{f^\varepsilon}\|_{L^2_{t,x}} \|f^\varepsilon\|_{L^2_{t,x,\xi}} \\ &\quad \le |\mathrm{supp}(\phi)|^{1/2} \|\phi\|_\infty \|\rho_{f^\varepsilon}\|_{L^2_t(L^{r(p)}_x)} \|f^\varepsilon\|_{L^2_{t,x,\xi}} \|K_\varepsilon - K\|_{L_x^{\frac{10p-6}{9p-7}}}. \end{aligned}$$

Here and below, $\|\cdot\|L^q_{t,x,\xi}$, $\|\cdot\|L^q_{t,x}$ and $L^{q_1}_t(L^{q_2}_x)$ denote norms of the function spaces $L^q((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, $L^q((0,T) \times \mathbb{R}^3_x)$ and $L^{q_1}((0,T);\ L^{q_2}(\mathbb{R}^3_x))$ respectively. Since

$$|\mathrm{supp}(\phi)|^{1/2} \|\phi\|_\infty \|\rho_{f^\varepsilon}\|_{L^2_t(L^{r(p)}_x)} \|f^\varepsilon\|_{L^2_{t,x,\xi}}$$

has an upper bound independent of ε, and since $\|K_\varepsilon - K\|_{L_x^{\frac{10p-6}{9p-7}}} \to 0$ as $\varepsilon \to 0$ due to $\frac{10p-6}{9p-7} < \frac{3}{2}$, [14] and [16], we obtain that $\|[(K_\varepsilon - K) \star \rho_{f^\varepsilon}] f^\varepsilon \phi\|_{L^1_{t,x,\xi}} \to 0$, which obviously implies (2.18). Secondly, we show

$$[(K \star (\rho_{f^\varepsilon} - \rho_f)] f^\varepsilon \to 0 \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3),\ \text{as } \varepsilon \to 0. \tag{2.19}$$

For any $\phi \in C_c^\infty((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, take a fixed $r > 0$ such that $\mathrm{supp}(\phi) \subset (0,T) \times B(0,r) \times B(0,r)$, denote $\rho_{(\phi f^\varepsilon)}(t,x) = \int_{\mathbb{R}^3} \phi f^\varepsilon d\xi$ and $\rho_{(\phi f)}(t,x) = \int_{\mathbb{R}^3} \phi f d\xi$. Then for any $R > r$

$$\begin{aligned} &\left| \int_0^T dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} [(K \star (\rho_{f^\varepsilon} - \rho_f)](t,x)(f^\varepsilon \phi)(t,x,\xi) dx d\xi \right| \\ &\le \left| \int_0^T dt \int_{\mathbb{R}^3} [(K \star \rho_{(\phi f^\varepsilon)}](t,x)(\rho_{f^\varepsilon} - \rho_f)(t,x) dx \right| \\ &\le \frac{1}{4\pi} \int_0^T dt \int_{\mathbb{R}^3} \left[\int_{B(0,r)} \frac{|\rho_{(\phi f^\varepsilon)}(t,y)|}{|x-y|^2} dy \right] |\rho_{f^\varepsilon} - \rho_f|(t,x) dx \\ &\le \frac{1}{4\pi} \int_0^T dt \int_{|x|>R} \left[\int_{B(0,r)} \frac{|\rho_{(\phi f^\varepsilon)}(t,y)|}{|x-y|^2} dy \right] |\rho_{f^\varepsilon} - \rho_f|(t,x) dx \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4\pi}\int_0^T dt\int_{|x|\le R}\left[\int_{B(0,r)}\frac{|\rho_{(\phi f^\varepsilon)}(t,y)|}{|x-y|^2}dy\right]|\rho_{f^\varepsilon}-\rho_f|(t,x)dx\\
&\le \frac{1}{4\pi(R-r)^2}\int_0^T dt\int_{\mathbb{R}^3}\left[\int_{B(0,r)}|\rho_{(\phi f^\varepsilon)}(t,y)|dy\right]|\rho_{f^\varepsilon}-\rho_f|(t,x)dx\\
&\quad+\frac{1}{4\pi}|B(0,r)|^{1/p'}\|\phi\|_\infty\sup_{0<t<T}\|f^\varepsilon(t)\|_p\\
&\quad\times\int_0^T dt\int_{|x|\le R}\left[\int_{B(0,r)}\frac{1}{|x-y|^2}dy\right]|\rho_{f^\varepsilon}-\rho_f|(t,x)dx\\
&\le \frac{T}{2\pi(R-r)^2}|B(0,r)|^{2/p'}\|\phi\|_\infty\|f_0\|_1\sup_{0<t<T}\|f^\varepsilon(t)\|_p\\
&\quad+2R|B(0,r)|^{1/p'}\|\phi\|_\infty\sup_{0<t<T}\|f^\varepsilon(t)\|_p\int_0^T dt\int_{|x|\le R}|\rho_{f^\varepsilon}-\rho_f|(t,x)dx.
\end{aligned}
$$

Letting $\varepsilon\to 0$ and $R\to\infty$ in succession and using (2.3) (2.14), we obtain (2.19). Finally, we show

$$
(K\star\rho_f)(f^\varepsilon-f)\to 0\quad\text{in } \mathcal{D}'((0,T)\times\mathbb{R}^3\times\mathbb{R}^3),\ \text{as } \varepsilon\to 0. \tag{2.20}
$$

In fact, the velocity averaging lemma implies that for any $\phi\in C_c^\infty((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$, a subsequence of $\rho_{(\phi f^\varepsilon)}(t,x)=\int_{\mathbb{R}^3}\phi f^\varepsilon d\xi$ converges to $\rho_{(\phi f)}(t,x)=\int_{\mathbb{R}^3}\phi f d\xi$ in $L^1_{\rm loc}([0,T]\times\mathbb{R}^3)$ as $\varepsilon\to 0$. Due to this result, the proof of (2.20) is the same as that of (2.19). From (2.17), (2.18), (2.19) and (2.20), we obtain the desired result (2.16).

Thanks to (2.12), (2.15) and (2.16), we can go to the limit $\varepsilon\to 0$ in the weak form of (2.1), (2.2). Actually, by Lemma 2.1 we have

$$
\begin{aligned}
&\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} f^\varepsilon(\partial_t\phi+\xi\cdot\nabla_x\phi+E^\varepsilon\cdot\nabla_\xi\phi)dxd\xi+\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\phi|_{t=0}dxd\xi\\
&\quad-\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} J(f^\varepsilon)\phi dxd\xi=0
\end{aligned}
$$

for any $\varepsilon>0$ and any $\phi(t,x,\xi)\in C_c^\infty([0,T)\times\mathbb{R}^3\times\mathbb{R}^3))$. Letting $\varepsilon\to 0$, we get for any $\phi(t,x,\xi)\in C_c^\infty([0,T)\times\mathbb{R}^3\times\mathbb{R}^3))$

$$
\begin{aligned}
&\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} f(\partial_t\phi+\xi\cdot\nabla_x\phi+E\cdot\nabla_\xi\phi)dxd\xi+\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\phi|_{t=0}dxd\xi\\
&\quad-\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} J(f)\phi dxd\xi=0.
\end{aligned}
$$

It implies that f is a global nonnegative solution to (1.1)–(1.5). Finally, Using (2.12), (2.14), (2.15) and (2.16), and passing to the limits in (2.3)–(2.5), we obtain (1.10), (1.11) and (1.12) (note that conservation of mass is obvious). □

3 Proof of Lemma 2.1

In order to show Lemma 2.1, we first consider a BGK equation with a given external field $E(t,x)$ which is assumed to be sufficiently regular, i.e., we discuss the following Cauchy problem:

$$\begin{aligned} &\partial_t f + \xi \cdot \nabla_x f + E(t,x) \cdot \nabla_\xi f = J(f), \quad (x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3,\ t \in [0,T], \\ &f(0,x,\xi) = f_0(x,\xi), \quad (x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3, \end{aligned} \tag{3.1}$$

where the external field $E(t,x) \in C([0,T]; C_b^1(\mathbb{R}^3))$ is fixed. We also assume that for some $\beta > 5$, $\gamma > 3$ there is a positive constant c such that $f_0(x,\xi) \geq \frac{c}{1+|x|^\gamma} \exp(-|\xi|^2)$ and $(1 + |\xi|^{\beta+\gamma}) f_0(x,\xi)$, $(1 + |\xi|^\beta)(1 + |x|^\gamma) f_0(x,\xi) \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ (this obviously implies that $(1+|\xi|^2) f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$). Under those assumptions, we know from the theory of ordinary differential equations that for any $(t,x,\xi) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$, the characteristic equations

$$\begin{cases} \dot{X}(s) = \Xi(s), & X(t) = x, \\ \dot{\Xi}(s) = E(s, X(s)), & \Xi(t) = \xi \end{cases} \tag{3.2}$$

of the first order partial differential equation (3.1) has a unique solution

$$Z(s,t;x,\xi) = (X(s,t;x,\xi),\ \Xi(s,t;x,\xi))$$

defined on $[0,T]$ such that

$$Z(s,t;x,\xi) \in C^1([0,T] \times [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3;\ \mathbb{R}^3 \times \mathbb{R}^3),$$

furthermore, for any fixed $s,t \in [0,T]$ it is a measure preserving homeomorphism from $\mathbb{R}^3 \times \mathbb{R}^3$ onto $\mathbb{R}^3 \times \mathbb{R}^3$. Hence, a nonnegative function $f \in L^\infty([0,T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ is a weak solution to (3.1) in distributional sense if and only if f satisfies the integral equation

$$f(t,x,\xi) = \exp(-t) f_0(Z(0,t;x,\xi)) + \int_0^t \exp(-(t-\tau)) M[f](\tau, Z(\tau,t;x,\xi)) d\tau, \tag{3.3}$$

or its equivalent form

$$f(t,x,\xi) = f_0(Z(0,t;x,\xi)) + \int_0^t J(f)(\tau, Z(\tau,t;x,\xi)) d\tau. \tag{3.4}$$

The characteristic flows have a series of estimates [32], for example, for any $s,t \in [0,T]$ and $(x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{aligned} &|\Xi(s,t;x,\xi)| \leq T\|E\|_\infty + |\xi|, && |\Xi(s,t;x,\xi) - \xi| \leq T\|E\|_\infty, \\ &|X(s,t;x,\xi)| \leq T^2\|E\|_\infty + |x| + T|\xi|, && |X(s,t;x,\xi) - x| \leq T^2\|E\|_\infty + T|\xi|, \end{aligned}$$

where $\|E\|_\infty$ is the $L^\infty((0,T) \times \mathbb{R}^3)$ norm of $E(t,x)$. Consequently, we get for any $s,t \in [0,T]$, $(x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{aligned} &|\xi| \leq T\|E\|_\infty + |\Xi(s,t;x,\xi)|, \\ &|x| \leq T^2\|E\|_\infty + T|\Xi(s,t;x,\xi)| + |X(s,t;x,\xi)|, \\ &(1 + |\xi|^\beta) \leq K_1(1 + |\Xi(s,t;x,\xi)|^\beta), \end{aligned}$$

and

$$\begin{aligned} (1 + |\xi|^\beta)(1 + |x|^\gamma) \leq\ & K_2(1 + |\Xi(s,t;x,\xi)|^{\beta+\gamma}) \\ & + K(1 + |\Xi(s,t;x,\xi)|^\beta)(1 + |X(s,t;x,\xi)|^\gamma), \end{aligned}$$

where $\beta,\gamma \geq 1$ and $K_1 = (2+2T\|E(\cdot,\cdot)\|_\infty)^\beta$, $K_2 = K_2(\beta,\gamma,T,\|E\|_\infty) = (3+3T+3T^2\|E\|_\infty)^\gamma K_1$. With the above assumptions and estimates, we can show the following result.

Lemma 3.1 *The Cauchy problem* (3.1) *has a unique nonnegative weak solution* $f \in C([0,T]; L^1(\mathbb{R}^3\times\mathbb{R}^3))$ *satisfying that there is a positive constant* $c_1 = c_1(T,\|E\|_\infty)$ *such that*

$$\rho_f(t,x) \geq \frac{c_1}{1+|x|^\gamma}, \quad t\in[0,T],\ x\in\mathbb{R}^3. \tag{3.5}$$

Furthermore, $\mathcal{E}_k(f)(t) \in L^1[0,T]$ *and for any* $p>1$ *there corresponds a positive constant* C_p' *independent of* ε *and* T *such that*

$$\|f(t)\|_1 = \|f_0\|_1, \qquad \|f(t)\|_p \leq \exp(C_p't)\|f_0\|_p, \quad t\in[0,T]. \tag{3.6}$$

Proof We carry out the proof in several steps.

Step 1. Lower bound estimates: We show that for any nonnegative solution f to (3.1) defined on $[0,T]$ there is a positive constant $c_1 = c_1(T)$ such that (3.5) holds. This is an easy consequence of (3.3), since for any $t\in[0,T]$ and $x\in\mathbb{R}^3$

$$\begin{aligned}
\rho_f(t,x) &\geq \exp(-t)\int_{\mathbb{R}^3} f_0(Z(0,t;x,\xi))d\xi \\
&\geq \exp(-T)\int_{\mathbb{R}^3}\frac{c}{1+|X(0,t;x,\xi)|^\gamma}\exp(-|\Xi(0,t;x,\xi)|^2)d\xi \\
&\geq \exp(-T-2T^2\|E\|_\infty^2)\int_{\mathbb{R}^3}\frac{c}{1+(T^2\|E\|_\infty+|x|+T|\xi|)^\gamma}\exp(-2|\xi|^2)d\xi \\
&\geq \frac{\exp(-T-2T^2\|E\|_\infty^2)}{(1+|x|^\gamma)}\int_{\mathbb{R}^3}\frac{c\exp(-2|\xi|^2)}{2^{\gamma-1}+2^{\gamma-1}(T^2\|E\|_\infty+T|\xi|)^\gamma}d\xi.
\end{aligned}$$

Taking

$$c_1 = \exp(-T-2T^2\|E\|_\infty^2)\int_{\mathbb{R}^3}\frac{c\exp(-2|\xi|^2)}{2^{\gamma-1}+2^{\gamma-1}(T^2\|E\|_\infty+T|\xi|)^\gamma}d\xi,$$

we get the desired estimate (3.5).

Step 2. Upper bound estimates: We shall need some weighted L^∞ norms introduced in [21], which are defined as follows

$$\mathbb{N}_\beta(f) = \sup_{(x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3}(1+|\xi|^\beta)|f(x,\xi)|,$$

$$\mathbb{N}_{\beta,\gamma}(f) = \sup_{(x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3}(1+|x|^\gamma)(1+|\xi|^\beta)|f(x,\xi)|.$$

We show that for any nonnegative solution f to (3.1) and a given $\beta>5$ there exist positive constants c_2, c_3 such that

$$\mathbb{N}_\beta(f(t)) \leq c_2\exp(c_3t)\mathbb{N}_\beta(f_0), \quad t\in[0,T], \tag{3.7}$$

$$\mathbb{N}_{\beta,\gamma}(f(t)) \le c_2 \exp(c_3 t)(\mathbb{N}_{\beta,\gamma}(f_0) + \mathbb{N}_{\beta+\gamma}(f_0)), \quad t \in [0, T]. \tag{3.8}$$

Multiplying both sides of (3.3) by $(1 + |\xi|^\beta)$, we obtain by the L^∞ estimates for local Maxwellians [21] and the estimates on characteristic flows deduced above

$$\begin{aligned}\mathbb{N}_\beta(f(t)) \le{}& K_1 \exp(-t)\mathbb{N}_\beta(f_0)\\ &+ K_1 C(\beta)\int_0^t \exp(-(t-\tau))\mathbb{N}_\beta(f(\tau))d\tau, \quad t \in [0, T],\end{aligned}$$

where $C(\beta) \ge 1$ is a positive constant dependent only upon β [21]. The Gronwall's lemma implies

$$\mathbb{N}_\beta(f(t)) \le K_1 \exp(K_1 C(\beta)t)\mathbb{N}_\beta(f_0), \quad t \in [0, T]. \tag{3.9}$$

Multiplying both sides of (3.3) by $(1 + |\xi|^\beta)(1 + |x|^\gamma)$, we get by a similar method that

$$\begin{aligned}\mathbb{N}_{\beta,\gamma}(f(t)) \le{}& K_2 \exp(-t)[\mathbb{N}_{\beta+\gamma}(f_0) + \mathbb{N}_{\beta,\gamma}(f_0)]\\ &+ K_2 C(\beta)\int_0^t \exp(-(t-\tau))[\mathbb{N}_{\beta+\gamma}(f(\tau)) + \mathbb{N}_{\beta,\gamma}(f(\tau))]d\tau, \quad t \in [0, T].\end{aligned}$$

This inequality and (3.9) obviously imply that

$$\begin{aligned}\mathbb{N}_{\beta,\gamma}(f(t)) \le{}& K_2 \exp(-t)[\mathbb{N}_{\beta+\gamma}(f_0) + \mathbb{N}_{\beta,\gamma}(f_0)]\\ &+ K_2 C(\beta)\exp(K_1 C(\beta)t)\mathbb{N}_{\beta+\gamma}(f_0)\\ &+ K_2 C(\beta)\int_0^t \exp(-(t-\tau))\mathbb{N}_{\beta,\gamma}(f(\tau))d\tau, \quad t \in [0, T].\end{aligned}$$

Then, the Gronwall's lemma gives

$$\begin{aligned}\mathbb{N}_{\beta,\gamma}(f(t)) \le{}& K_2(1 + C(\beta) + K_2 C(\beta)^2)\exp(\max\{K_1, K_2\}C(\beta)t)\\ &\times [\mathbb{N}_{\beta+\gamma}(f_0) + \mathbb{N}_{\beta,\gamma}(f_0)], \quad t \in [0, T].\end{aligned} \tag{3.10}$$

From (3.9)and (3.10), we obtain the desired estimates (3.7) and (3.8) by taking

$$c_2 = K_2(1 + C(\beta) + K_2 C(\beta)^2) \quad \text{and} \quad c_3 = K_2 C(\beta).$$

Step 3. Existence and uniqueness: It is well known that the weighted Lebesgue space $X = L^1(\mathbb{R}^3 \times \mathbb{R}^3; (1 + |\xi|^2)dxd\xi)$ with norm

$$\|f\|_X = \int_{\mathbb{R}^3\times\mathbb{R}^3} |f(x,\xi)|(1 + |\xi|^2)dxd\xi$$

is a Banach space. Then the metric space

$$\begin{aligned}X(T) = \{&f \in C([0, T]; X):\\ &f \text{ is nonnegative and verifies (3.5), (3.7) and (3.8)}\}\end{aligned}$$

with metric $\|f_1 - f_2\|_T = \max_{t\in[0,T]} \|f_1(t) - f_2(t)\|_X$ is a closed subset of the Banach space $C([0, T]; X)$. It has been proved in [21] that there exists a positive constant L_0 such that for any $f_1, f_2 \in X(T)$

$$\|M[f_1](t) - M[f_2](t)\|_X \le L_0 \|f_1(t) - f_2(t)\|_X, \quad t \in [0, T]. \tag{3.11}$$

Now, we define a nonlinear operator F on $X(T)$ by

$$(Ff)(t, x, \xi) = \exp(-t) f_0(Z(0, t; x, \xi)) + \int_0^t \exp(-(t-\tau)) M[f](\tau, Z(\tau, t; x, \xi)) d\tau,$$

then (3.5), (3.7) and (3.8) show that F maps $X(T)$ into itself. On the other hand, it follows from (3.11) that for any $f_1, f_2 \in X(T)$

$$\|(Ff_1)(t) - (Ff_2)(t)\|_X \le L \int_0^t \|f_1(s) - f_2(s)\|_X ds, \quad t \in [0, T],$$

where $L = 2(1 + T^2\|E\|_\infty^2) L_0$. It implies that for any nature number n

$$\|(F^n f_1) - (F^n f_2)\|_T \le \frac{(LT)^n}{n!} \|f_1 - f_2\|_T.$$

Consequently, for n large enough the operator F^n is a contraction from X_T into itself. So, the Banach fixed point theorem ensures that the operator F has a unique fixed point in X_T. Obviously the fixed point is the unique solution to (3.1) verifying (3.5).

Step 4. Proof of other estimates: The estimate $\mathcal{E}_k(f)(t) \in L^1[0, T]$ follows directly from (3.8). To prove (3.6), taking L^p norms on both sides of (3.3), we obtain

$$\|f(t)\|_p \le \exp(-t)\|f_0\|_p + \int_0^t \exp(-(t-s))\|M[f](s)\|_p ds, \quad t \in [0, T].$$

Then, we use the L^p estimate $\|M[f]\|_p \le C(p)\|f\|_p$ of local Maxwellians (it is valid for nonnegative functions $f(x, \xi)$ satisfying $f \in L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ for any $p > 1$ and $(1 + |\xi|^2) f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$)(see [33], Corollary 2.3), we further obtain

$$\|f(t)\|_p \le \exp(-t)\|f_0\|_p + C(p) \int_0^t \exp(-(t-s))\|f(s)\|_p ds, \quad t \in [0, T].$$

Then, the Gronwall's lemma implies that

$$\|f(t)\|_p \le \exp((C(p) - 1)t)\|f_0\|_p, \quad t \in [0, T].$$

Letting $C'_p = C(p) - 1 \ge 0$, we get the second estimate in (3.6). The first estimate in (3.6) is implied by the same method and $\|M[f](t)\|_1 = \|f(t)\|_1$ for any $t \in [0, T]$. □

Having proved Lemma 3.1, we are in a position to construct a sequence of approximate solutions to the Cauchy problem (2.1), (2.2). To this end, for any nature number j we introduce a closed convex subset $\mathcal{C}_j$ of the positive cone $L^1_{\text{loc}}([0, \infty); L^1(\mathbb{R}^3))_+$ of the space $L^1_{\text{loc}}([0, \infty); L^1(\mathbb{R}^3))$ as follows

$$\mathcal{C}_j = \{\rho(t, x) \in L^1_{\text{loc}}([0, \infty);\ L^1(\mathbb{R}^3))_+ : \rho(t, x) \le j \text{ and } \rho(t, x) = 0 \text{ for } |x| > j \text{ or } t > j\},$$

then we define a continuous mapping Ψ_j from $L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^3))_+$ into $\mathcal{C}_j$ by

$$(\Psi_j\rho)(t,x) = \begin{cases} \rho(t,x), & (t,x) \in [0,j] \times B(0,j) \text{ and } \rho(t,x) \le j; \\ j, & (t,x) \in [0,j] \times B(0,j) \text{ and } \rho(t,x) > j; \\ 0, & t > j \text{ or } x \in B(0,j)^c. \end{cases}$$

Here, $B(0,j) = \{x \in \mathbb{R}^3 : |x| \le j\}$ and $B(0,j)^c = \{x \in \mathbb{R}^3 : |x| > j\}$. We denote by $J_j(t)$ the one dimensional regularizer with regularizing radius $1/j$, then the approximate Cauchy problems for (2.1), (2.2) are constructed as follows

$$\partial_t f_j^\varepsilon + \xi \cdot \nabla_x f_j^\varepsilon + E_j^\varepsilon \cdot \nabla_\xi f_j^\varepsilon = J(f_j^\varepsilon), \qquad f_j^\varepsilon(0,x,\xi) = f_{0,j}(x,\xi), \tag{3.12}$$

$$E_j^\varepsilon(t,x) = \gamma\{K_\varepsilon(\cdot) \star_x [J_j(\cdot) \star_t (\Psi_j \rho_{f_j^\varepsilon})](t,\cdot)\}(x), \tag{3.13}$$

$$(x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3,\ t \in [0,j].$$

Here, $f_{0,j}(x,\xi)$ is defined by

$$f_{0,j}(x,\xi) = \varphi_j(x,\xi)\max\{f_0(x,\xi), j\} + \frac{1}{j}\frac{\exp(-|\xi|^2)}{(1+|x|^\gamma)},$$

in which $\gamma > 3$ is a fixed number and φ_j is the usual cutoff function such that $0 \le \varphi_j \le 1$, $\varphi_j(x,\xi) = 0$ for $|x|^2 + |\xi|^2 \ge j^2$, and $\lim_{j\to\infty}\varphi(x,\xi) = 1$. It is obvious that

$$\lim_{j\to\infty}\int_{\mathbb{R}^3\times\mathbb{R}^3}(1+|\xi|^2)|f_{0,j} - f_0|(x,\xi)dx d\xi = 0, \qquad \lim_{j\to\infty}\|f_{0,j} - f_0\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = 0,$$

and for any $\beta > 0$

$$f_{0,j}(x,\xi) \ge \frac{1}{j}\frac{\exp(-|\xi|^2)}{(1+|x|^\gamma)}, \qquad \mathbb{N}_{\beta,\gamma}(f_{0,j}) < \infty,\ \mathbb{N}_{\beta+\gamma}(f_{0,j}) < \infty.$$

As a consequence, for any $j = 1,2,\ldots$, $f_{0,j}$ as an initial datum satisfies all assumptions of Lemma 3.1.

Lemma 3.2 *Under the above assumptions, for any fixed $j = 1,2,\ldots$ and any fixed $\varepsilon > 0$, the Cauchy problem* (3.12), (3.13) *has a nonnegative weak solution $f_j^\varepsilon \in L^\infty([0,j]; L^1(\mathbb{R}^3\times\mathbb{R}^3))$. Furthermore, for any $p > 1$ there corresponds a positive constant C'_p independent of ε and j such that*

$$\|f_j^\varepsilon(t)\|_1 = \|f_{0,j}\|_1, \qquad \|f_j^\varepsilon(t)\|_p \le \exp(C'_p t)\|f_{0,j}\|_p, \quad t \in [0,j], \tag{3.14}$$

$$\mathcal{E}_k(f_j^\varepsilon)(t) \le [\mathcal{E}_k(f_{0,j}) + \varepsilon^{-2}\|f_{0,j}\|_1^3]\exp(t), \quad t \in [0,j]. \tag{3.15}$$

Proof According to Lemma 3.1, for any given $\tilde{\rho}(t,x) \in \mathcal{C}_j$ the Cauchy problem for BGK equation

$$\begin{aligned} &\partial_t f + \xi\cdot\nabla_x f + E_{\tilde{\rho}}(t,x)\cdot\nabla_\xi f = J(f), \quad (x,\xi) \in \mathbb{R}^3\times\mathbb{R}^3,\ t \in [0,j], \\ &f(0,x,\xi) = f_{0,j}(x,\xi), \quad (x,\xi) \in \mathbb{R}^3\times\mathbb{R}^3 \end{aligned} \tag{3.16}$$

has a unique nonnegative solution $f \in C([0, j]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ satisfying that $\mathcal{E}_k(f)(t) \in L^1[0, j]$ and that for any $p > 1$ there corresponds a positive constant C'_p independent of ε and j such that

$$\|f(t)\|_1 = \|f_{0,j}\|_1, \qquad \|f(t)\|_p \le \exp(C'_p t)\|f_{0,j}\|_p, \quad t \in [0, j], \tag{3.17}$$

where $E_{\tilde\rho}(t, x) = \gamma\{K_\varepsilon(\cdot) \star_x [J_j(\cdot) \star_t \tilde\rho](t, \cdot)\}(x) \in C([0, j]; C_b^\infty(\mathbb{R}^3))$ and $\|E_{\tilde\rho}(t, \cdot)\|_\infty \le \varepsilon^{-1}\|\tilde\rho(t, \cdot)\|_1$ for $t \in [0, j]$. On the other hand, it is easy to show that [32]

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_k(f)(t) &= 2\int_{\mathbb{R}^3\times\mathbb{R}^3} \xi \cdot E_{\tilde\rho}(t, x) f(t, x, \xi)dxd\xi \\ &\le \mathcal{E}_k(f)(t) + \|f_{0,j}\|_1 \||E_{\tilde\rho}(t)|\|_\infty^2 \\ &\le \mathcal{E}_k(f)(t) + \varepsilon^{-2}\|f_{0,j}\|_1^3, \quad t \in [0, j].\end{aligned} \tag{3.18}$$

Then, the Gronwall's lemma gives that

$$\mathcal{E}_k(f)(t) \le [\mathcal{E}_k(f_{0,j}) + \varepsilon^{-2}\|f_{0,j}\|_1^3]e^t, \quad t \in [0, j]. \tag{3.19}$$

Due to these results we define a mapping F from $\mathcal{C}_j$ into itself by $F\tilde\rho = \Psi_j \rho_f$. At present, we suppose that the operator F is continuous and maps $\mathcal{C}_j$ into a compact subset of itself (i.e., F is completely continuous), then the Schauder's theorem ensures that F has a fixed point in $\mathcal{C}_j$. We denote the fixed point by ρ_j^ε and the corresponding solution to (3.16) by f_j^ε, then $\rho_j^\varepsilon = \Psi_j \rho_{f_j^\varepsilon}$ and consequently $E_{\tilde\rho}(t, x) = \gamma\{K_\varepsilon(\cdot) \star_x [J_j(\cdot) \star_t (\Psi_j \rho_{f_j^\varepsilon})](t, \cdot)\}(x)$. This result and estimates (3.17) and (3.19) imply that f_j^ε is a nonnegative solution to (3.12), (3.13) such that it verifies (3.14) and (3.15).

To finish the proof, it is sufficient to show that F is a completely continuous operator from $\mathcal{C}_j$ into itself. Firstly, we show that F maps $\mathcal{C}_j$ into a compact subset of itself. We denote by $f^{\tilde\rho}$ the unique solution to (3.16) associate with $\tilde\rho \in \mathcal{C}_j$. It follows from (3.17), (3.19), the L^p estimates for local Maxwellians [33], and $\|E_{\tilde\rho}(t, \cdot)\|_\infty \le \varepsilon^{-1}\|\tilde\rho(t, \cdot)\|_1 \le \varepsilon^{-1} j|B(0, j)|$ that $\{f^{\tilde\rho} : \tilde\rho \in \mathcal{C}_j\}$ and $\{-E_{\tilde\rho}(t, x) \cdot \nabla_\xi f^{\tilde\rho} + J(f^{\tilde\rho}) : \tilde\rho \in \mathcal{C}_j\}$ are bounded subsets in $L^2((0, j) \times \mathbb{R}^3 \times \mathbb{R}^3)$ and $L^2((0, j) \times \mathbb{R}_x^3; H^{-1}(\mathbb{R}_\xi^3))$ respectively. Hence, the velocity averaging lemma [10] and the estimate (3.19) imply that $\{\rho_{f^{\tilde\rho}} : \tilde\rho \in \mathcal{C}_j\}$ is relatively compact in $L^1_{\mathrm{loc}}([0, j] \times \mathbb{R}^3)$, as a consequence we know that $\{\Psi_j \rho_{f^{\tilde\rho}} : \tilde\rho \in \mathcal{C}_j\}$ is relatively compact in $\mathcal{C}_j$. That is to say that F is compact.

Next, we show that F is continuous. Let $\tilde\rho, \tilde\rho_n \in \mathcal{C}_j$ $(n = 1, 2, \ldots)$ and $\tilde\rho_n \to \tilde\rho$ in $\mathcal{C}_j$ as $n \to \infty$, we have to show that $F\tilde\rho_n \to F\tilde\rho$ in $\mathcal{C}_j$ as $n \to \infty$. It is equivalent to show that any subsequence of $F\tilde\rho_n$ has a subsequence that converges in $\mathcal{C}_j$ to $F\tilde\rho$. Since Ψ_j is continuous, it is also sufficient to show that any subsequence of $\rho_{f^{\tilde\rho_n}}$ has a subsequence that converges in $L^1([0, j] \times B(0, j))$ to $\rho_{f^{\tilde\rho}}$. On the other hand, the last step implies that any subsequence of $\rho_{f^{\tilde\rho_n}}$ does have a subsequence that converges in $L^1([0, j] \times B(0, j))$. Hence, it remains to show that the limit must be $\rho_{f^{\tilde\rho}}$. Without loss of generality, we may assume that $\rho_{f^{\tilde\rho_n}}$ converges in $L^1([0, j] \times B(0, j))$ to some $\rho \in L^1([0, j] \times B(0, j))$, then we prove $\rho = \rho_{f^{\tilde\rho}}$. Actually, we may also assume that $f^{\tilde\rho_n}$ converges weakly in $L^p((0, j) \times \mathbb{R}^3 \times \mathbb{R}^3)$ to some nonnegative function $f \in L^1 \cap L^p((0, j) \times \mathbb{R}^3 \times \mathbb{R}^3)$ for $1 < p < \infty$. Let $h^n = J(f^{\tilde\rho_n})$ and $g^n = -E_{\tilde\rho_n} f^{\tilde\rho_n}$, then it follows from (3.16) that

$$\begin{aligned}&\partial_t f^{\tilde\rho_n} + \xi \cdot \nabla_x f^{\tilde\rho_n} = h^n + \nabla_\xi g^n, \quad (x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3,\ t \in [0, j], \\ &f^{\tilde\rho_n}(0, x, \xi) = f_{0,j}(x, \xi), \quad (x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3.\end{aligned} \tag{3.20}$$

Similar to the proof of Theorem 1.1, (3.20) and the velocity moments lemma (Lemma 2.4) imply that for any $R>0$

$$\begin{aligned}\int_0^j dt\int_{B(0,R)\times\mathbb{R}^3_\xi}&|\xi|^{5/2}f^{\tilde\rho_n}dxd\xi\\ &\le K(1+R)\int_0^j dt\int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi}[(1+|\xi|)|g^n|+|\xi|^{3/2}|h^n|]dxd\xi\\ &\quad+K(1+R)\int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi}[|\xi|^{3/2}+|\xi|^2]|f_{0,j}|dxd\xi,\end{aligned}\tag{3.21}$$

where K is a positive constant independent of n. Due to the above estimates, it is easy to show that the left hand side of (3.21) has a finite upper bound independent of n. This result and the velocity averaging lemma imply that $\int_{\mathbb{R}^3}|\xi|^r f^{\tilde\rho_n}(t,x,\xi)d\xi$ is compact in $L^1_{\rm loc}([0,j]\times\mathbb{R}^3)$ for any fixed $r\in[0,5/2)$. Consequently, we have for any $R>0$ when $n\to\infty$ (up to a subsequence)

$$\begin{aligned}
&f^{\tilde\rho_n}\to f, && \text{in } L^p((0,j)\times\mathbb{R}^3\times\mathbb{R}^3),\\
&\rho_{f^{\tilde\rho_n}}\to\rho_f, && \text{in } L^1((0,j)\times B(0,R)),\\
&\rho_{f^{\tilde\rho_n}}(|u_{f^{\tilde\rho_n}}|^2+3\theta_{f^{\tilde\rho_n}})\to\rho_f(|u_f|^2+3\theta_f), && \text{in } L^1((0,j)\times B(0,R)),\\
&\rho_{f^{\tilde\rho_n}}u_{f^{\tilde\rho_n}}\to\rho_f u_f, && \text{in } L^1((0,j)\times B(0,R))^3,\\
&J(f^{\tilde\rho_n})\to J(f), && \text{in } L^1((0,j)\times B(0,R)\times\mathbb{R}^3_\xi).
\end{aligned}\tag{3.22}$$

On the other hand, it is easy to show that

$$E_{\tilde\rho_n}(t,x)\to E_{\tilde\rho}(t,x),\quad \text{in } L^\infty((0,j)\times\mathbb{R}^3).\tag{3.23}$$

By (3.22), (3.23), we can go to the limits in distributional sense in (3.20) and obtain that

$$\begin{aligned}&\partial_t f+\xi\cdot\nabla_x f+E_{\tilde\rho}(t,x)\cdot\nabla_\xi f=J(f),\quad (x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3,\ t\in[0,j],\\ &f(0,x,\xi)=f_{0,j}(x,\xi),\quad (x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3.\end{aligned}\tag{3.24}$$

By Lemma 3.1, $f^{\tilde\rho}$ is the unique solution of (3.24). So, $f=f^{\tilde\rho}$ and $\rho_f=\rho_{f^{\tilde\rho}}$. This result and (3.22) give that $\rho_{f^{\tilde\rho_n}}\to\rho_{f^{\tilde\rho}}$ in $L^1([0,j]\times B(0,j))$ as $n\to\infty$. Hence, F is continuous from $\mathcal{C}_j$ into itself. □

Proof of Lemma 2.1 Based on Lemma 3.2, the proof of the lemma is similar to and much easier than that of Theorem 1.1. So, we only give a very sketchy proof of it by pointing out key estimates which is sufficient for using velocity averaging lemma and velocity moments lemma. In the following, we always assume that the initial datum f_0 verifies the conditions given in Lemma 2.1 and we also use the notation $f_{0,j}$ $(j=1,2,\dots)$ given before Lemma 3.2. We will show that a subsequence of solutions $\{f_j^\varepsilon : j=1,2,\dots\}$ to (3.12), (3.13) given in Lemma 3.2 converges weakly to a solution of the Cauchy problem (2.1), (2.2). Actually, we can restrict ourselves to any bounded time interval $[0,T]$ since a diagonal method enable us to extend the solution to all positive times. Let $p\ge 2$, then for any $j\ge T$, f_j^ε verifies (3.14), (3.15) for all $t\in[0,T]$. Furthermore, it is obvious that $\|E_{\rho_{f_j^\varepsilon}}(t,\cdot)\|_\infty\le\varepsilon^{-1}\|\rho_{f_j^\varepsilon}(t,\cdot)\|_1=\varepsilon^{-1}\|f_{0,j}\|_1$ for $t\in[0,T]$. Since $\mathcal{E}_k(f_{0,j})$, $\|f_{0,j}\|_1$ and $\|f_{0,j}\|_p$ have finite upper bounds independent of $j=1,2,\dots$, the velocity averaging lemma and the velocity moments lemma imply that there exists a subsequence of $\{f_j^\varepsilon : j = 1, 2, \ldots\}$ (denoted by itself) and a nonnegative function f^ε such that for any $R > 0$ as $j \to \infty$

$$\begin{aligned}
&f_{0,j} \to f_0, && \text{in } L^1(\mathbb{R}^3 \times \mathbb{R}^3),\\
&f_j^\varepsilon \to f, && \text{in } L^p((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3),\\
&\rho_{f_j^\varepsilon} \to \rho_f, && \text{in } L^1((0,T) \times B(0,R)),\\
&\rho_{f_j^\varepsilon}(|u_{f_j^\varepsilon}|^2 + 3\theta_{f_j^\varepsilon}) \to \rho_f(|u_f|^2 + 3\theta_f), && \text{in } L^1((0,T) \times B(0,R)),\\
&\rho_{f_j^\varepsilon} u_{f_j^\varepsilon} \to \rho_f u_f, && \text{in } L^1((0,T) \times B(0,R))^3,\\
&J(f_j^\varepsilon) \to J(f), && \text{in } L^1((0,T) \times B(0,R) \times \mathbb{R}^3_\xi),\\
&E_{\rho_{f_j^\varepsilon}}(t,x) \to \gamma\{K_\varepsilon(\cdot) \star_x \rho_f(t,\cdot)\}(x), && \text{in } L^\infty((0,T) \times \mathbb{R}^3)^3.
\end{aligned}$$

Due to these results, we can go to the limits in the weak form of (3.12), (3.13) and obtain that

$$\begin{aligned}
&\int_0^T dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} f^\varepsilon \left(\partial_t \phi + \xi \cdot \nabla_x \phi + E^\varepsilon \cdot \nabla_\xi \phi\right) dx d\xi + \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0 \phi|_{t=0} dx d\xi\\
&\quad - \int_0^T dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} J(f^\varepsilon)\phi dx d\xi = 0
\end{aligned}$$

for any test function $\phi(t,x,\xi) \in C_c^\infty([0,T) \times \mathbb{R}^3 \times \mathbb{R}^3))$, where $E^\varepsilon(t,x) = \gamma\{K_\varepsilon(\cdot) \star_x \rho_f(t,\cdot)\}(x)$. Consequently, f^ε is a weak solution on $[0,T)$ to (2.1) (2.2).

To finish the proof, it is sufficient to show that f^ε satisfies (2.3), (2.4) and (2.5). (2.3) follows directly from (3.14). On the other hand, since $|\xi|^2$ is a collision invariant of the BGK operator $J(f)$, (2.4) can be obtained by a similar method used in [16] and [32]. Now, we prove (2.5). In fact, by the definition of $\mathcal{E}_{p,\varepsilon}(f^\varepsilon)(t)$, Hölder inequality and Lemma 2.2 we obtain

$$\begin{aligned}
|\mathcal{E}_{p,\varepsilon}(f^\varepsilon)(t)| &\le \frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_{f^\varepsilon}(t,x)\rho_{f^\varepsilon}(t,y)}{|x-y|} dx dy \le C\|\rho_{f^\varepsilon}(t,\cdot)\|_{6/5}^2\\
&\le C\|\rho_{f^\varepsilon}(t,\cdot)\|_1^{\frac{7p-9}{6(p-1)}} \|\rho_{f^\varepsilon}(t,\cdot)\|_{r(p)}^{\frac{5p-3}{6(p-1)}}\\
&\le C[C(p)]^{\frac{5p-3}{6(p-1)}} \|f_0\|_1^{\frac{7p-9}{6(p-1)}} \|f^\varepsilon(t)\|_p^{\frac{p}{3(p-1)}} \||\xi|^2 f^\varepsilon(t)\|_1^{\frac{1}{2}}\\
&\le C[C(p)]^{\frac{5p-3}{6(p-1)}} \|f_0\|_1^{\frac{7p-9}{6(p-1)}} [\exp(\tilde{C}_p t)\|f_0\|_p]^{\frac{p}{3(p-1)}} [\mathcal{E}_k(f^\varepsilon)(t)]^{\frac{1}{2}}\\
&= \sqrt{\tilde{M}} \exp\left(\frac{p}{3(p-1)} \tilde{C}_p t\right) [\mathcal{E}_k(f^\varepsilon)(t)]^{\frac{1}{2}},
\end{aligned}$$

where $\sqrt{\tilde{M}} = C[C(p)]^{\frac{5p-3}{6(p-1)}} \|f_0\|_1^{\frac{7p-9}{6(p-1)}} \|f_0\|_p^{\frac{p}{3(p-1)}}$. Inserting this inequality into (2.4), we get

$$\mathcal{E}_k(f^\varepsilon)(t) \le [\mathcal{E}_k(f_0) + |\mathcal{E}_{p,\varepsilon}(f_0)|] + \sqrt{\tilde{M}} \exp\left(\frac{p}{3(p-1)} \tilde{C}_p t\right) [\mathcal{E}_k(f^\varepsilon)(t)]^{\frac{1}{2}},$$

which obviously implies (2.5). □

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